9) Prove that $\operatorname{div}(r^n \vec{r}) = (n+3) r^n$. Hence show that $\frac{\vec{r}}{r^3}$ is solenoidal.

w.k.T
$\operatorname{div} \vec{r} = 3$

Proof: Consider $\operatorname{div}(r^n \vec{r})$,

$\Rightarrow \phi(r) = r^n \quad , \quad \vec{F} = \vec{r}$

$\left[\operatorname{div}(\phi \vec{F}) = \phi \operatorname{div} \vec{F} + \nabla \phi \cdot \vec{F} \right]$

$\nabla \phi(r) = \frac{d\phi}{dr} \frac{\vec{r}}{r}$

$$
\begin{aligned}
\operatorname{div}(r^n \vec{r}) &= r^n \operatorname{div} \vec{r} + \nabla r^n \cdot \vec{r} \\
&= r^n\, 3 + \frac{d}{dr}(r^n) \frac{\vec{r}}{r} \cdot \vec{r} \\
&= 3r^n + n r^{n-1} \frac{\vec{r}}{r} \cdot \vec{r} \\
&= 3r^n + n r^{n-2} (\vec{r} \cdot \vec{r}) \\
&= 3r^n + n r^{n-2}\, r^2 \\
&= 3r^n + n r^{n-\cancel{2}+\cancel{2}} \\
&= 3r^n + n r^n \\
&= r^n (3+n) \\
&= r^n (n+3)
\end{aligned}
$$

$\vec{r} \cdot \vec{r} = r^2$

To show $\frac{\vec{r}}{r^3}$ is solenoidal, $\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = 0$

W.k.T $\quad div(r^n \vec{r}) = (n+3) r^n \rightarrow (1)$

$$div\left(\frac{\vec{r}}{r^3}\right) \Rightarrow div(r^{-3}\vec{r})$$

put $n = -3$ in (1)

$$div(r^{-3}\vec{r}) = (-3+3) r^{-3}$$
$$= 0$$

$\therefore \frac{\vec{r}}{r^3}$ is a solenoidal vector.

10) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r \neq 0$, show that

$$grad\left(div \frac{\vec{r}}{r}\right) = -\frac{2\vec{r}}{r^3}$$

Solution: Consider $grad\left(div \frac{\vec{r}}{r}\right)$,

To find $div\left(\frac{\vec{r}}{r}\right) \Rightarrow div(r^{-1}\vec{r})$

W.k.T $\quad div(r^n \vec{r}) = (n+3) r^n$

$$div(r^{-1}\vec{r}) = (-1+3) r^{-1}$$
$$= 2r^{-1}$$
$$= 2/r$$

$$grad\left(div \frac{\vec{r}}{r}\right) = grad\left(\frac{2}{r}\right)$$

$$= 2 \text{ grad}\left(\frac{1}{r}\right) \qquad \nabla \phi(r) = \frac{d\phi}{dr} \cdot \frac{\vec{r}}{r}$$

$$= 2 \nabla\left(\frac{1}{r}\right)$$

$$= 2 \frac{d}{dr}\left(\frac{1}{r}\right) \frac{\vec{r}}{r}$$

$$= 2\left(-\frac{1}{r^2}\right) \frac{\vec{r}}{r}$$

$$= -\frac{2\vec{r}}{r^3}$$

iii) If $\vec{u} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and $\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

Show that $\vec{u} \times \vec{v}$ is a solenoidal vector.

<u>Solution:</u> Given, $\vec{u} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

$\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$

To find $\vec{u} \times \vec{v}$,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$$

$$\vec{u} \times \vec{v} = (y^3x - z^3x)\hat{i} - (x^3y - z^3y)\hat{j} + (x^3z - y^3z)\hat{k}$$

To show $\vec{u} \times \vec{v}$ is solenoidal,

$$\text{div}(\vec{u} \times \vec{v}) = 0$$

$$div(\vec{u}\times\vec{v}) = \frac{\partial}{\partial x}(y^3x - z^3x) + \frac{\partial}{\partial y}(z^3y - x^3y) + \frac{\partial}{\partial z}(x^3z - y^3z)$$

$$= \cancel{y^3} - \cancel{z^3} + \cancel{z^3} - \cancel{x^3} + \cancel{x^3} - \cancel{y^3}$$

$$= 0$$

$\therefore$ $\vec{u}\times\vec{v}$ is a solenoidal vector.

12) Prove that $div\left\{r^2 \nabla\left(\frac{1}{r^3}\right)\right\} = 0$

Proof: Consider, $div\left\{r^2 \nabla\left(\frac{1}{r^3}\right)\right\}, \rightarrow (1)$

To find $\nabla\left(\frac{1}{r^3}\right)$

$$\nabla\left(\frac{1}{r^3}\right) = \frac{d}{dr}\left(\frac{1}{r^3}\right)\frac{\vec{r}}{r}$$

$$= -\frac{3}{r^4}\frac{\vec{r}}{r}$$

$$= -\frac{3\vec{r}}{r^5} \rightarrow (2)$$

Sub, (2) in (1)

$$div\left\{\cancel{r^2}\,\frac{-3\vec{r}}{\cancel{r^5}}\right\} = -3\, div\left\{\frac{\vec{r}}{r^3}\right\}$$

$$= -3\, div\left\{r^{-3}\vec{r}\right\}$$

$= -3\ (-3+3)\ r^{-3}$

Using $\text{div}\,(r^n \vec{r}) = (n+3)\,r^n$

$= -3\ (0)$

$= \underline{\underline{0}}$

The Laplacian:

In electrostatics, the gradient of the electric potential is a scalar multiple of the electric field intensity and the divergence of the electric field intensity is related to the charge density. For this and other reasons it is convenient to introduce a single operator that is the composite of the two operators grad and div. This operator is called the Laplacian.

The Laplacian of a scalar field ϕ is defined to be $\text{div}\,(\text{grad}\,\phi)$.

In notation, $\nabla\cdot(\nabla\phi) \Rightarrow \nabla^2\phi$

Laplacian $(\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

The differential operator ∇^2 is called the Laplacian operator.

Harmonic Functions:

A function satisfying Laplace equation is called a Harmonic function.

$\Rightarrow \nabla^2\phi = 0$.

Problems:

1) If $\phi = x^2 - y^2 + 4z$ Show that $\nabla^2\phi = 0$

Solution: Given $\phi = x^2 - y^2 + 4z$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial^2\phi}{\partial x^2} = 2$$

$$\frac{\partial\phi}{\partial y} = -2y, \quad \frac{\partial^2\phi}{\partial y^2} = -2$$

$$\frac{\partial \phi}{\partial z} = 4 \quad , \quad \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\nabla^2 \phi = 2 + (-2) + 0$$
$$= 2 - 2 + 0$$
$$= \underline{\underline{0}}$$

2) Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$

$\underline{\text{proof}}$: $\nabla^2\left(\frac{1}{r}\right) = \operatorname{div}\left(\operatorname{grad}\left(\frac{1}{r}\right)\right)$

$$\operatorname{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\frac{\vec{r}}{r}$$
$$= -\frac{1}{r^2}\frac{\vec{r}}{r}$$
$$= -\frac{\vec{r}}{r^3}$$

$$\operatorname{div}\left(\frac{-\vec{r}}{r^3}\right) = -\operatorname{div}\left(\frac{\vec{r}}{r^3}\right)$$
$$= -\operatorname{div}\left(r^{-3}\,\vec{r}\right)$$
$$= -d(-3+3)\, r^{-3}$$
$$= 0 //$$

$\operatorname{div}(r^n \vec{r})$
$= \underline{(n+3)\, r^n}$

3) If n is a non-zero constant, show that

$$\nabla^2 r^n = n(n+1)\, r^{n-2}$$

Solution: Consider $\nabla^2 (r^n) = \text{div}\{\text{grad}(r^n)\}$

$$\text{grad}(r^n) = \nabla r^n = \frac{d}{dr}(r^n)\,\frac{\vec{r}}{r}$$

$$= n\, r^{n-1}\,\frac{\vec{r}}{r}$$

$$= n\, r^{n-2}\,\vec{r}$$

$$\text{div}\{n\, r^{n-2}\vec{r}\} = n\ \text{div}\{r^{n-2}\,\vec{r}\}$$

$$= n\left[(n-2+3)\, r^{n-2}\right]$$

$$= n\,[n+1]\, r^{n-2}$$

$$= n(n+1)\, r^{n-2} \,//$$

4) Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$
where $r^2 = x^2+y^2+z^2$

Solution; Consider $\nabla^2 f(r) = \text{div}\left(\text{grad}\, f(r)\right)$

$$\text{grad}\,[f(r)] = \nabla f(r) = \frac{d}{dr}(f(r))\,\frac{\vec{r}}{r}$$

$$\nabla f(r) = f'(r)\,\frac{\vec{r}}{r}$$

$$\operatorname{div}\left[\frac{f'(r)}{r}\,\vec{r}\right] = \frac{f'(r)}{r}\operatorname{div}\vec{r} + \nabla\frac{f'(r)}{r}\cdot\vec{r}$$

$$= \frac{f'(r)}{r}\,3 + \nabla\frac{f'(r)}{r}\cdot\vec{r} \rightarrow 1$$

$$\nabla\frac{f'(r)}{r} = \frac{d}{dr}\left(\frac{f'(r)}{r}\right)\frac{\vec{r}}{r}$$

$$= \left[f'(r)\left(\frac{-1}{r^2}\right) + f''(r)\frac{1}{r}\right]\frac{\vec{r}}{r}$$

$$= \left[-\frac{f'(r)}{r^2} + \frac{f''(r)}{r}\right]\frac{\vec{r}}{r}$$

$$= \left[\frac{f''(r)\,\vec{r}}{r^2} - \frac{f'(r)\,\vec{r}}{r^3}\right] \rightarrow 2$$

Sub 2 in 1

$$= 3\,\frac{f'(r)}{r} + \left[\frac{f''(r)\,\vec{r}}{r^2} - \frac{f'(r)\,\vec{r}}{r^3}\right]\cdot\vec{r}$$

$$= 3\,\frac{f'(r)}{r} + \left[\frac{f''(r)(\vec{r}\cdot\vec{r})}{r^2} - \frac{f'(r)(\vec{r}\cdot\vec{r})}{r^3}\right]$$

$$= \frac{3f'(r)}{r} + \left[\frac{f''(r)\,\cancel{r^2}}{\cancel{r^2}} - \frac{f'(r)\,\cancel{r^2}}{r^{\cancel{3}}}\right]$$

$$= \frac{3f'(r)}{r} + \left[f''(r) - \frac{f'(r)}{r}\right]$$

$$= 3\frac{f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= 2\frac{f'(r)}{r} + f''(r)$$

H.W

1) S.T $\nabla^2 r^2 = 6$

2) S.T $\nabla^2(\log r) = \frac{1}{r^2}$